Transactions on Computer Science and Intelligent Systems Research
ISSN: 2960-1800, eISSN: 2960-2238 | Volume 13
AMMSAC 2026

A conjecture on the absolute separability of eight by eight positive semi-definite matrices

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Abstract. The separability problem, a long-standing open challenge in mathematics and quantum information theory. In this work, we propose and investigate a conjecture as a sub-question of the separability problem. Specifically, we investigate whether the form $I_8 - xx^*$ is separable for any eight-dimensional unit vector x . This is actually the so-called absolute separability problem. For studying the conjecture, we classify the vectors in a finitely-dimensional bipartite Hilbert space, and construct their equivalence classes in terms of the so-called Schmidt rank, under both product general linear and product linear groups. Then we construct seven equivalence classes of x, using which we partially prove the conjecture. Our results highlight the the complexity of the difference between bipartite and tripartite spaces.

Keywords: Separability, Positive Semidefinite Matrix, Kronecker Product.

1. Introduction

Determining the existence of decomposition of a positive semidefinite matrix on a bipartite Hilbert space into the convex sum of Kronecker product of positive semidefinite matrices is the separability problem in quantum information theory [1]. The matrix with such a decomposition is called separable [2]. The separability problem has been shown to be an NP-hard problem in computational complexity, as well as a new phenomenon related to both quantum physics [3] and operator algebra such as positive maps [4] and completely positive maps [5]. The hardness of the problem is due to the fact that it is usually not operational to find the exact decomposition into the Kronecker product of semidefinite positive matrices [6]. The separability problem has attracted significant attention in recent decades. Mathematically, the absolute separability problem — a special case of the separability problem — has received considerable attention. It states that under the action of a global unitary matrix, some separable positive semidefinite matrix ρ remains separable [7]. This ρ is called absolutely separable. It has been shown that if a $2 \times n$ matrix ρ satisfies certain conditions, then ρ is absolutely separable [8]. The partial transpose turns out to be a key mathematical tool in the absolute separability problem [9], [10]. The spectra condition of matrices has also been studied recently [11]. However, so far, the absolute separability problem for tripartite Hilbert space remains mathematically unexplored.

In this paper, we investigate the simplest tripartite scenario, namely the case of 8×8 positive semidefinite matrices. To start the investigation, we review some basics from linear algebra, such as matrix and Kronecker product. Next, we review the basic group theory including the general linear (GL) group, unitary group and product GL (PGL) group and product unitary (PU) group. We use them to simplify the vectors in a finitely-dimensional bipartite Hilbert space, such as $C^2 \otimes C^2$, $C^2 \otimes C^n$, and $C^m \otimes C^n$. We exactly construct their equivalence classes in terms of the so-called Schmidt rank, under both PGL and PU groups. We apply them to the one and two-dimensional subspaces of bipartite Hilbert space. Then we study the tripartite Hilbert space $C^2 \otimes C^2 \otimes C^2$, on which some linear operators can be chosen as the 8×8 positive semidefinite matrices M. We study trace-one M of rank one, i.e., $M = xx^*$ with a unit vector x, and find out seven equivalent classes of x under the PGL group. Then we propose our conjecture, namely $\rho = I_8 - xx^*$ is always absolutely separable. We managed to prove that the conjecture works when x is PGL equivalent to the first five classes.

Due to the hardness of class six and seven, we investigated some partial case, which also support the conjecture.

2. Basics

In this section, we introduce the basic knowledge and facts used in this paper.

2.1. Matrix and linear algebra

We will use the notation $e_i^{(m)}$, $i = 1,\dots,m$ to represent a vector of the standard orthonormal basis of $\mathbb{C}^m$.

2.2. Kronecker Product

2.2.1. Definition

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. The Kronecker product of A and B, denoted by $A \otimes B$, is defined as the $mp \times nq$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B. \end{bmatrix}, \tag{1}$$

Where a_{ij} denotes the elements of matrix A, and each block $a_{ij}B$ is the scalar multiplication of B by a_{ij}.

2.2.2. Properties

The Kronecker product in [eq: kronecker] satisfies several useful algebraic properties:

1. Associativity:

$$\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}. \tag{2}$$

2. Left Distributivity:

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) \quad = (\mathbf{A} \otimes \mathbf{B}) + (\mathbf{A} \otimes \mathbf{C}). \tag{3}$$

3. Right Distributivity:

$$(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} \quad = (\mathbf{A} \otimes \mathbf{C}) + (\mathbf{B} \otimes \mathbf{C}). \tag{4}$$

4. Scalar Multiplication: For scalar a,

$$a \otimes \mathbf{A} = \mathbf{A} \otimes a = a\mathbf{A}. \tag{5}$$

5. Scalar products: For scalars a and b,

$$a\mathbf{A} \otimes b\mathbf{B} = ab(\mathbf{A} \otimes \mathbf{B}). \tag{6}$$

6. Mixed product property: If $\mathbf{AC}$ and $\mathbf{BD}$ are defined, then

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}). \tag{7}$$

7. Transpose and conjugate:

$$(\mathbf{A} \otimes \mathbf{B})^\top = \mathbf{A}^\top \otimes \mathbf{B}^\top \tag{8}$$

$$(\mathbf{A} \otimes \mathbf{B})^* = \mathbf{A}^* \otimes \mathbf{B}^* \tag{9}$$

2.3. Basic Group Theory

2.3.1. Group

A *group* is an algebraic structure consisting of a set G together with a binary operation $\cdot : G \times G \to G$ that satisfies the following axioms:

1. Closure: For all $a, b \in G$, the result of the operation $a \cdot b$ is also an element of G, that is,

$$a \cdot b \in G. \tag{10}$$

This ensures that combining any two elements of the group using the group operation always yields another element in the group.

2. Associativity: For all $a, b, c \in G$, the group operation is associative:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c). \tag{11}$$

Associativity allows us to write products without ambiguity regarding the order of operations.

3. Identity element: There exists an element $e \in G$, known as the *identity element*, such that for all $a \in G$,

$$e \cdot a = a = a \cdot e. \tag{12}$$

The identity element acts as a neutral element in the group, leaving other elements unchanged when combined.

4. Inverses: For each element $a \in G$, there exists an element $a^{-1} \in G$, called the *inverse* of a, such that

$$a \cdot a^{-1} = e = a^{-1} \cdot a. \tag{13}$$

The existence of inverses guarantees that every element in the group has an inverse under the group operation.

2.4. Matrix Groups

2.4.1. General Linear Group (GL group)

The *general linear group* of degree n over a field $\mathbb{F}$, denoted $\mathrm{GL}(n, \mathbb{F})$, is defined as the set of all invertible $n \times n$ matrices with entries from $\mathbb{F}$:

$$\mathrm{GL}(n, \mathbb{F}) \coloneqq \{A \in \mathbb{F}^{n \times n} \mid \det A \neq 0\}. \tag{14}$$

The group operation is standard matrix multiplication. The group satisfies the following key properties:

Closure: If $A, B \in \mathrm{GL}(n, \mathbb{F})$, then the product $AB \in \mathrm{GL}(n, \mathbb{F})$ as

$$\det(AB) = \det A \cdot \det B \neq 0. \tag{15}$$

Hence, the set is closed under matrix multiplication.

Associativity: If $A, B, C \in \mathrm{GL}(n, \mathbb{F})$, then

$$(AB)C = A(BC), \tag{16}$$

Since matrix multiplication is associated with all matrices of the same size. Thus, associativity holds in $\mathrm{GL}(n, \mathbb{F})$.

Identity: The identity matrix $I_n \in \mathrm{GL}(n, \mathbb{F})$ satisfies $\det I_n = 1 \neq 0$. Moreover, for any matrix $A \in \mathrm{GL}(n, \mathbb{F})$,

$$AI_n = I_n A = A. \tag{17}$$

Thus, I_n acts as the identity element of the group.

Inverse: Since $\det A \neq 0$ for all $A \in \mathrm{GL}(n, \mathbb{F})$, the inverse matrix A^{-1} exists and also belongs to the group. Additionally,

$$\det(A^{-1}) = \frac{1}{\det A} \neq 0, \tag{18}$$

Ensuring closure under inversion.

The general linear group is a central object in linear algebra and plays a foundational role in group theory and representation theory.

2.4.2. Unitary Group

The *unitary* group of degree n, denoted $\mathrm{U}(n)$, is defined as the group of all $n \times n$ complex matrices U that preserve the standard Hermitian inner product. Formally,

$$\mathrm{U}(n) \coloneqq \{U \in \mathbb{C}^{n \times n} \mid U^* U = I_n\}, \tag{19}$$

Where U^* denotes the conjugate transpose of U. The group operation is matrix multiplication. The unitary group satisfies the following properties:

Closure: If $U_1, U_2 \in \mathrm{U}(n)$, then the product $U_1 U_2 \in \mathrm{U}(n)$, since

$$(U_1 U_2)^* (U_1 U_2) = U_2^* U_1^* U_1 U_2 = U_2^* I_n U_2 = I_n. \tag{20}$$

Associativity: If $A, B, C \in \mathrm{U}(n)$, then

$$(AB)C = A(BC), \tag{21}$$

Because matrix multiplication is associative for all matrices of the same size. Therefore, associativity holds in $\mathrm{U}(n)$.

Identity: The identity matrix I_n satisfies $I_n^* I_n = I_n$, and thus belongs to $\mathrm{U}(n)$. It serves as the neutral element under matrix multiplication.

Inverse: For any $U \in \mathrm{U}(n)$, the inverse is given by $U^{-1} = U^*$. This follows from the condition $U^* U = I_n$, implying that U is unitary and its conjugate transpose serves as its inverse.

Unitary matrices are important in quantum mechanics and signal processing due to their norm-preserving properties.

2.4.3. Orthogonal Group

The *orthogonal group* of degree n over a field $\mathbb{F}$, denoted $\mathrm{O}(n, \mathbb{F})$, is defined as the set of all $n \times n$ matrices $Q \in \mathbb{F}^{n \times n}$ satisfying

$$Q^\top Q = Q Q^\top = I_n, \tag{22}$$

Where $Q^\top$ denotes the transpose of Q. This condition ensures that the columns (and rows) of Q form an orthonormal set with respect to the standard inner product.

Closure: If $Q_1, Q_2 \in \mathrm{O}(n, \mathbb{F})$, then the product $Q_1 Q_2 \in \mathrm{O}(n, \mathbb{F})$ as

$$(Q_1 Q_2)^\top (Q_1 Q_2) = Q_2^\top Q_1^\top Q_1 Q_2 = Q_2^\top I_n Q_2 = I_n. \tag{23}$$

Associativity: If $A, B, C \in \mathrm{O}(n)$, then

$$(AB)C = A(BC), \tag{24}$$

Since matrix multiplication is associated with all matrices of the same size. Hence, associativity holds in $\mathrm{O}(n)$.

Identity: The identity matrix I_n clearly satisfies $I_n^\top I_n = I_n$, and thus belongs to $\mathrm{O}(n, \mathbb{F})$.

Inverse: For any $Q \in \mathrm{O}(n, \mathbb{F})$, the transpose $Q^\top$ is the inverse of Q, since

$$Q Q^\top = Q^\top Q = I_n. \tag{25}$$

Therefore, $Q^{-1} = Q^\top$, and the group is closed under inversion.

Orthogonal matrices are widely used in numerical linear algebra due to their numerical stability and preservation of vector norms.

2.4.4. Product General Linear Group (PGL group)

The product *general linear group* of degree (m, n) in the field $\mathbb{F}$, denoted $\mathrm{PGL}(m, n, \mathbb{F})$, is defined as

$$\mathrm{PGL}(m, n, \mathbb{F}) = \{A \otimes B \mid A \in \mathrm{GL}(m, \mathbb{F}), \quad B \in \mathrm{GL}(n, \mathbb{F})\}. \tag{26}$$

Lemma 1. *Let $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_m\}$ be two sets of m linearly independent vectors in $\mathbb{C}^n$. Then, there exists a matrix $A \in \mathrm{GL}(n, \mathbb{C})$ such that $Ax_i = y_i$ for all i, establishing a one-to-one correspondence between X and Y.*

Proof. We can extend each set to a basis of $\mathbb{C}^n$ by choosing vectors $x_{m+1}, \dots, x_n$ and $y_{m+1}, \dots, y_n$ such that

$\{x_1, \dots, x_n\}$ forms a basis for $\mathbb{C}^n$,

$\{y_1, \dots, y_n\}$ forms another basis for $\mathbb{C}^n$.

Let $\mathcal{X} = [x_1 \ \cdots \ x_n]$ and $\mathcal{Y} = [y_1 \ \cdots \ y_n]$. We define

$$A = \mathcal{Y}\mathcal{X}^{-1}, \tag{27}$$

Which satisfies $Ax_i = y_i$ for all i. □

2.5. Lemmas

Lemma 2. *If matrix H satisfies that $H = H^* = \begin{bmatrix} a & \bar{b} \\ b & c \end{bmatrix}$, where $a, c \in \mathbb{R}$, then $H \succcurlyeq 0$ is equivalent to $a, c \geq 0$ and $ac - |b|^2 \geq 0$.*

Proof. Suppose H has two eigenvalues λ_1 and λ_2. Recall that $\lambda_1 + \lambda_2 = \mathrm{tr}(H) = a + c$, $\lambda_1\lambda_2 = \det(H) = ac - bb^* = ac - |b|^2$. Since $a, c \geq 0$ and $ac - |b|^2 \geq 0$, $\lambda_1 + \lambda_2 \geq 0$, $\lambda_1\lambda_2 \geq 0$. Thus, $\lambda_1, \lambda_2 \geq 0$, H is positive semidefinite.

Lemma 3. *If matrices $A_1, A_2, \cdots A_n$ are all positive semidefinite matrices, then the block matrix*

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{bmatrix} \succcurlyeq 0. \tag{28}$$

Proof. Assume each matrix A_i has dimension $n_i \times n_i$. For any vector $x \in \mathbb{C}^n$, where $n = \sum_{i=1}^{k} n_i$, we write it in block form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}, \quad \text{with} x_i \in \mathbb{C}^{n_i}. \tag{29}$$

Then,

$$x^* M x = \sum_{i=1}^{k} x_i^* A_i x_i. \tag{30}$$

Since each A_i is positive semidefinite, we have $x_i^* A_i x_i \geq 0$, so the sum is non-negative:

$$x^* M x \geq 0. \tag{31}$$

Therefore, $M \succcurlyeq 0$.

Lemma 4. Suppose *the matrix H satisfies that $H = H^*$ and $H \succcurlyeq 0$, and P is a matrix. Then $PHP^* \succcurlyeq 0$.*

Proof. For any nonzero vector $x \in \mathbb{C}^n$, define $y = P^* x$. Then we have:

$$x^*(PHP^*)x = (P^*x)^* H (P^*x) = y^* H y. \tag{32}$$

Because $H \succcurlyeq 0$, it follows that $y^* H y \geq 0$ for all $y \in \mathbb{C}^n$. Therefore,

$$x^*(PHP^*)x = y^*Hy \geq 0. \tag{33}$$

Hence, $PHP^* \succcurlyeq 0$.

A

3. Vector Simplification via PGL Group: Dimension one

3.1. Problem Statement

Given a vector $\mathbf{v} \in \mathbb{C}^m \otimes \mathbb{C}^n$, we seek a transformation $B \in \mathrm{GL}(m,\mathbb{C}) \otimes \mathrm{GL}(n,\mathbb{C})$ that simplifies $\mathbf{v}$—ideally minimizing the number of nonzero components in $B \cdot \mathbf{v}$.

3.2. Case Study: $\mathbb{C}^2 \otimes \mathbb{C}^2$

3.2.1. Case 1: Matrix is Invertible

Assume $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$, and select $B = I \otimes B_0 = \begin{bmatrix} B_0 & \mathbf{0} \\ \mathbf{0} & B_0 \end{bmatrix}$. We aim to transform $\mathbf{v}$ into the simplified form,

$$B \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{34}$$

This is equivalent to the condition

$$B_0 \cdot \begin{bmatrix} v_1 & v_3 \\ v_2 & v_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{35}$$

If the matrix $M = \begin{bmatrix} v_1 & v_3 \\ v_2 & v_4 \end{bmatrix}$ is invertible, then we may choose

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot M^{-1} = M^{-1}. \tag{36}$$

3.2.2. Case 2: Matrix is Not Invertible

Suppose that the matrix is not invertible. Assume the first column is a scalar multiple of the second

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = k \begin{bmatrix} v_3 \\ v_4 \end{bmatrix}. \tag{37}$$

We consider a transformation of the form

$$B = \begin{bmatrix} 1 & -k \\ c_1 & c_2 \end{bmatrix} \otimes \begin{bmatrix} v_4 & -v_3 \\ c_3 & c_4 \end{bmatrix}, \tag{38}$$

Where $c_1, c_2, c_3, c_4 \in \mathbb{C}$ are arbitrary complex parameters.

Computing the Kronecker product gives

$$B = \begin{bmatrix} v_4 & -v_3 & -kv_4 & kv_3 \\ c_3 & c_4 & -kc_3 & -kc_4 \\ c_1 v_4 & -c_1 v_3 & c_2 v_4 & -c_2 v_3 \\ c_1 c_3 & c_1 c_4 & c_2 c_3 & c_2 c_4 \end{bmatrix}. \tag{39}$$

Apply B to $\mathbf{v} = \begin{bmatrix} kv_3 \\ kv_4 \\ v_3 \\ v_4 \end{bmatrix}$,

$$B \cdot \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (kc_1c_3 + c_2c_3)v_3 + (kc_1c_4 + c_2c_4)v_4 \end{bmatrix}. \tag{40}$$

To ensure that $B \in \mathrm{GL}(2, \mathbb{C}) \otimes \mathrm{GL}(2, \mathbb{C})$, we must choose values of c_1, c_2, c_3, c_4 such that

$$\det \begin{bmatrix} 1 & -k \\ c_1 & c_2 \end{bmatrix} = c_2 + kc_1 \neq 0, \tag{41}$$

And

$$\det \begin{bmatrix} v_4 & -v_3 \\ c_3 & c_4 \end{bmatrix} = c_4 v_4 + c_3 v_3 \neq 0. \tag{42}$$

Furthermore, if we select k be $\frac{1 - c_2c_3v_3 - c_2c_4v_4}{c_1c_3v_3 + c_1c_4v_4}$, then

$$\mathrm{B} \cdot \mathrm{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{43}$$

3.3. Case Study: $\mathbb{C}^2 \otimes \mathbb{C}^n$

Assume

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{2n} \end{bmatrix}, \tag{44}$$

And define the matrix

$$V = \begin{bmatrix} v_1 & v_{n+1} \\ v_2 & v_{n+2} \\ \vdots & \vdots \\ v_n & v_{2n} \end{bmatrix}. \tag{45}$$

3.3.1. Case 1: $\mathbf{rank}(\boldsymbol{V}) = \mathbf{2}$

Select

$$B = I \otimes B_0, \quad \text{where} B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} (V^\top V)^{-1} V^\top. \tag{46}$$

Since $V^\top V$ is invertible (because V has full rank),

$$B \cdot V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} (V^\top V)^{-1} V^\top V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix}. \tag{47}$$

Thus,

$$B \cdot v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{48}$$

3.3.2. Case 2: $\mathbf{rank}(\boldsymbol{V}) = \mathbf{1}$

Suppose V is rank-one, that is, the first column is a scalar multiple of the second,

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = k \begin{bmatrix} v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{2n} \end{bmatrix}. \tag{49}$$

Next we observe

$$v = \begin{bmatrix} k \\ 1 \end{bmatrix} \otimes \begin{bmatrix} v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{2n} \end{bmatrix}. \tag{50}$$

We define

$$B = \begin{bmatrix} -1 & k \\ c_1 & c_2 \end{bmatrix} \otimes P, \tag{51}$$

Where $c_1, c_2 \in \mathbb{C}$ are arbitrary and P is a matrix on $\mathbb{C}^n$. Then

$$B \cdot v = \left(\begin{bmatrix} -1 & k \\ c_1 & c_2 \end{bmatrix} \cdot \begin{bmatrix} k \\ 1 \end{bmatrix} \right) \otimes \left(P \cdot \begin{bmatrix} v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{2n} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ kc_1 + c_2 \end{bmatrix} \otimes \left(P \cdot \begin{bmatrix} v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{2n} \end{bmatrix} \right). \tag{52}$$

From the lemma, there exists $P_0 \in \mathrm{GL}(n, \mathbb{C})$ such that

$$P_0 \cdot \begin{bmatrix} v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{2n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{53}$$

Hence

$$B \cdot v = \begin{bmatrix} 0 \\ kc_1 + c_2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ kc_1 + c_2 \end{bmatrix}. \tag{54}$$

To ensure that $B \in \mathrm{GL}(2, \mathbb{C}) \otimes \mathrm{GL}(n, \mathbb{C})$, we have

$$\det \begin{bmatrix} -1 & k \\ c_1 & c_2 \end{bmatrix} = -c_2 - kc_1 \neq 0. \tag{55}$$

Equivalently, we require

$$c_2 + kc_1 \neq 0. \tag{56}$$

3.4. General Case: $\mathbb{C}^m \otimes \mathbb{C}^n$

Lemma 5. *Under the PGL group, any vector* $v \in \mathbb{C}^m \otimes \mathbb{C}^n$ *can be simplified to* $\sum_{i=1}^{s} e_i^{(m)} \otimes e_i^{(n)}$, $s \in N$. *We say that* s *is the Schmidt rank of vector* v. *It is invariant under PGL equivalence.*

Proof. **Part 1: v can be simplified to**

$$\sum_{i=1}^{s} e_i^{(m)} \otimes e_i^{(n)} \tag{57}$$

Let $v \in \mathbb{C}^m \otimes \mathbb{C}^n$, where $m \leq n$. Then we may write

$$v = a_1 \otimes b_1 + \cdots + a_m \otimes b_m, \tag{58}$$

Where $a_1, a_2, \ldots, a_m \subset \mathbb{C}^m$, and

$$span\{a_1, a_2, \ldots, a_m\} = \mathbb{C}^m. \tag{59}$$

We assume $I_m = \left(e_1^{(m)}, e_2^{(m)}, \ldots, e_m^{(m)}\right)$ is the standard basis of $\mathbb{C}^m$. Let $b_{k_1}, \ldots, b_{k_s}$ be linearly independent vectors in $\mathbb{C}^n$, such that

$$b_{k_{s+1}}, \ldots, b_{k_m} \in \mathrm{span}\{b_{k_1}, \ldots, b_{k_s}\}, \quad 1 \leq s \leq n. \tag{60}$$

Then there exists a permutation operation P such that

$$P: \{k_1, k_2 \cdots k_m\}\, \{1,2, \cdots s, k_{s+1}, k_{s+2}, k_m\}. \tag{61}$$

Thus,

$$P \otimes I_n : v \rightarrow \sum_{i=1}^{s} a_i \otimes b_i + \sum_{i=s+1}^{m} a_i \otimes b_i. \tag{62}$$

Each b_i for $i > s$ can be expressed as a linear combination. Hence, for all $i > s$, we have

$$b_i = \sum_{k=1}^{s} \alpha_{i,k}\, b_k, \tag{63}$$

With $\alpha_{i,k} \in \mathbb{C}$. Substituting back into v, we have

$$\begin{aligned} v \quad &= \sum_{k=1}^{s} a_k \otimes b_k + \sum_{i=s+1}^{m} a_i \otimes \left(\sum_{k=1}^{s} \alpha_{i,k}\, b_k\right) \\ &= \sum_{k=1}^{s} a_k \otimes b_k + \sum_{k=1}^{s} \left(\sum_{i=s+1}^{m} \alpha_{i,k}\, a_i\right) \otimes b_k \\ &= \sum_{k=1}^{s} \left(a_k + \sum_{i=s+1}^{m} \alpha_{i,k}\, a_i\right) \otimes b_k. \end{aligned} \tag{64}$$

We define new vectors for $1 \leq k \leq s$ as follows: $a'_k := a_k + \sum_{i=s+1}^{m} \alpha_{i,k}\, a_i$. For the sake of convenience, from Lemma 1 we can consider multiplying a matrix that turns the basis $\{a_1, a_2, a_3 \cdots a_m\}$ to $\{e_1, e_2, e_3 \cdots e_m\}$. Then we have

$$\{a'_1, a'_2, \cdots a'_s\} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \alpha_{s+1,1} \\ \vdots \\ \alpha_{m,1} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \alpha_{s+1,2} \\ \vdots \\ \alpha_{m,2} \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \\ \alpha_{s+1,s} \\ \vdots \\ \alpha_{m,s} \end{pmatrix} \tag{65}$$

Since $\{a'_1, a'_2, \dots, a'_s\}$ is a set of linearly independent vectors, by Lemma 1, we can multiply the matrices appropriately so that a'_i is mapped to $e_i^{(m)}$ and b_i is mapped to $e_i^{(n)}$. Consequently, the vector v can be simplified to

$$\sum_{i=1}^{s} e_i^{(m)} \otimes e_i^{(n)}. \tag{66}$$

Part 2: Schmidt rank of vector v is an invariant under PGL equivalence
We assume

$$v = a_1 \otimes b_1 + \cdots + a_m \otimes b_m. \tag{67}$$

We define the matrix

$$A = [a_1, a_2, \cdots a_m] \tag{68}$$

And

$$B = [b_1, b_2, \cdots b_m]. \tag{69}$$

Using eq. (59), the Schmidt rank of vector v is just the rank of matrix B, without loss of generality, let's assume $\mathrm{Col}(B) = \mathrm{span}\{b_1, b_2 \cdots b_s\}$. If we consider two matrices $P \in \mathrm{GL}(m, \mathbb{C}), Q \in \mathrm{GL}(n, \mathbb{C})$

$$(P \otimes Q) \cdot v = Pa_1 \otimes Qb_1 + Pa_2 \otimes Qb_2 + \cdots + Pa_m \otimes Qb_m, \tag{70}$$

The only thing we need to show is

$$\mathrm{rank}(B) = \mathrm{rank}(QB). \tag{71}$$

On one hand, we have

$$\mathrm{rank}(QB) \leq \mathrm{rank}(B) = s. \tag{72}$$

On the other hand, consider a matrix

$$B' = [b_1 \quad b_2 \quad \cdots \quad b_s \quad v_{s+1} \quad v_{s+2} \quad \cdots \quad v_n], \tag{73}$$

Where the vectors $v_{s+1}, v_{s+2}, \dots, v_n$ are chosen such that

$$\mathrm{span}\{b_1, b_2, \dots, b_s, v_{s+1}, v_{s+2}, \dots, v_n\} = \mathbb{R}^n. \tag{74}$$

That is, B' is a full-rank matrix in $\mathbb{R}^{n\times n}$. Since $Q \in \mathrm{GL}(m, \mathbb{C})$, we have $\det(Q) \neq 0$, and because $\det(B') \neq 0$, it follows that

$$\det(QB') = \det(Q)\det(B') \neq 0. \tag{75}$$

Thus, QB' is also a full-rank matrix. In particular, the first s columns of QB', namely $Qb_1, Qb_2, \dots, Qb_s$, must be linearly independent. Therefore,

$$\mathrm{rank}(QB) \geq \dim(\mathrm{span}\{Qb_1, Qb_2, \dots, Qb_s\}) = s. \tag{76}$$

Combining both results, we conclude that

$$\mathrm{rank}(QB) = s. \tag{77}$$

So, we proved that Schmidt rank of vector v is an invariant under PGL equivalence.

We have simplified a vector $v \in \mathbb{C}^m \otimes \mathbb{C}^n$ into the form

$$u_s = \sum_{i=1}^{s} e_i^{(m)} \otimes e_i^{(n)}. \tag{78}$$

Now, we would like to explore some properties of u_s. There do not exist matrices A and B such that

$$A \otimes B \in \mathrm{PGL}(m, n, \mathbb{C}) \tag{79}$$

And

$$(A \otimes B)u_p = u_q. \tag{80}$$

Lemma 6. For $p \neq q$, *the two vectors*

$$u_p = \sum_{i=1}^{p} e_i^{(m)} \otimes e_i^{(n)}, \tag{81}$$

$$u_q = \sum_{i=1}^{q} e_i^{(m)} \otimes e_i^{(n)} \tag{82}$$

are not equivalent under the action of $\mathrm{PGL}(m, n, \mathbb{C})$.

Proof. Suppose such A and B exist. Then we have

$$(A \otimes B)\left(\sum_{i=1}^{p} e_i^{(m)} \otimes e_i^{(n)}\right) = \sum_{i=1}^{q} e_i^{(m)} \otimes e_i^{(n)}. \tag{83}$$

Expanding the left-hand side gives

$$\sum_{i=1}^{p}\left(Ae_i^{(m)}\right) \otimes \left(Be_i^{(n)}\right) = \sum_{i=1}^{q} e_i^{(m)} \otimes e_i^{(n)}. \tag{84}$$

By taking the partial transpose, this leads to

$$\sum_{i=1}^{p}\left(Ae_i^{(m)}\right)\left(Be_i^{(n)}\right)^\top = \sum_{i=1}^{q} e_i^{(m)} \left(e_i^{(n)}\right)^\top. \tag{85}$$

Since

$$\sum_{i=1}^{p}\left(Ae_i^{(m)}\right)\left(e_i^{(n)}\right)^\top B^\top = A\left(\sum_{i=1}^{p} e_i^{(m)} \left(e_i^{(n)}\right)^\top\right)B^\top, \tag{86}$$

We obtain

$$A\left(\sum_{i=1}^{p} e_i^{(m)} \left(e_i^{(n)}\right)^\top\right)B^\top = \sum_{i=1}^{q} e_i^{(m)} \left(e_i^{(n)}\right)^\top. \tag{87}$$

However, $\mathrm{rank}(_{\mathrm{LHS}}) = p$ while $\mathrm{rank}(_{\mathrm{RHS}}) = q$, contradicting the assumption that $p \neq q$. Thus, no such matrices A and B exist.

4. Vector Simplification via PGL Group: Dimension 2

Lemma 7. *Let* u *and* v *be two linearly independent vectors in* $\mathbb{C}^2 \otimes \mathbb{C}^2$. *Then the subspace* $\mathrm{span}\{u, v\}$ *contains a product vector.*

Proof. We may assume that

$$v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad u = \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}. \tag{88}$$

Case 1: Suppose $ac - bd = 0$ or $ef - gh = 0$.

Without loss of generality, assume $ac - bd = 0$. Then there exists $x \in \mathbb{C}$ such that $c = ax$ and $d = bx$. Hence,

$$v = \begin{bmatrix} a \\ b \\ ax \\ bx \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix} \otimes \begin{bmatrix} a \\ b \end{bmatrix}, \tag{89}$$

Which is a product vector. Since $v \in \text{span}\{u, v\}$, the subspace contains a product vector.
Case 2: Suppose $ac - bd \neq 0$ and $ef - gh \neq 0$.

Consider a linear combination $w = k_1 u + k_2 v$ for some $k_1, k_2 \in \mathbb{C}$. Then

$$w = \begin{bmatrix} ak_1 + ek_2 \\ bk_1 + fk_2 \\ ck_1 + gk_2 \\ dk_1 + hk_2 \end{bmatrix}. \tag{90}$$

We want w to be product vectors, i.e., $w = x \otimes y$ for some $x, y \in \mathbb{C}^2$. This is equivalent to the condition

$$\frac{ck_1 + gk_2}{ak_1 + ek_2} = \frac{dk_1 + hk_2}{bk_1 + fk_2}, \tag{91}$$

Provided the denominators are nonzero. Cross-multiplying gives

$$(ck_1 + gk_2)(bk_1 + fk_2) = (dk_1 + hk_2)(ak_1 + ek_2), \tag{92}$$

Which simplifies to the quadratic

$$(bc - ad)k_1^2 + (bg + cf - ah - de)k_1 k_2 + (fg - eh)k_2^2 = 0. \tag{93}$$

Setting $k_1 = 1$, we get

$$(fg - eh)k_2^2 + (bg + cf - ah - de)k_2 + (bc - ad) = 0. \tag{94}$$

Since this equation has a solution in $\mathbb{C}$, we conclude that such a linear combination $u + k_2 v$ is a product vector, and therefore $\text{span}\{u, v\}$ contains a product vector.

4.1. Equivalence Classes of $\mathbb{C}^8$ under the Action of $\mathbf{PGL(2,2,2,\mathbb{C})}$

We say that two vectors $u, v \in \mathbb{C}^8$ are equivalent under the action of $\text{PGL}(2,2,2,\mathbb{C})$ if there exists a matrix $M \in \text{PGL}(2,2,2,\mathbb{C})$, such that $Mu = v$. In particular, we suppose $M = A \otimes B \otimes C$, where $A, B, C \in \text{GL}(2,\mathbb{C})$.

Lemma 8. *There are exactly seven equivalence classes of* $\mathbb{C}^8$ *under the action of* $\text{PGL}(2,2,2,\mathbb{C})$*, namely [eq:class 1]- [eq:class 7].*

Proof. **Equivalence Class 1:**

$$(A \otimes B \otimes C) \cdot \mathbf{0} \equiv \mathbf{0}. \tag{95}$$

So, the zero vector forms its own equivalence class.

Equivalence Class 2:

$$e_1 \otimes e_1 \otimes e_1 \tag{96}$$

$$(A \otimes B \otimes C) \cdot (e_1 \otimes e_1 \otimes e_1) = (Ae_1 \otimes Be_1 \otimes Ce_1). \tag{97}$$

Suppose

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{98}$$

Then

$$Ae_1 = \begin{bmatrix} a_1 \\ a_3 \end{bmatrix}, \quad Be_1 = \begin{bmatrix} b_1 \\ b_3 \end{bmatrix}, \quad Ce_1 = \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} \tag{99}$$

And hence

$$(A \otimes B \otimes C) \cdot (e_1 \otimes e_1 \otimes e_1) = \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ c_3 \end{bmatrix}. \tag{100}$$

Thus, all vectors of the form $p \otimes q \otimes r$ with $p, q, r \in \mathbb{C}^2$ are equivalent to $e_1 \otimes e_1 \otimes e_1$ under the action of $\mathrm{PGL}(2,2,2,\mathbb{C})$.

Equivalence Class 3:

$$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) \tag{101}$$

$$(A \otimes B \otimes C) \cdot \big(e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)\big) = Ae_1 \otimes (Be_1 \otimes Ce_1 + Be_2 \otimes Ce_2). \tag{102}$$

Again, using the matrix forms above, we compute

$$Ae_1 \otimes (Be_1 \otimes Ce_1 + Be_2 \otimes Ce_2) = \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \otimes \left(\begin{bmatrix} b_1 \\ b_3 \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} + \begin{bmatrix} b_2 \\ b_4 \end{bmatrix} \otimes \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} \right) \tag{103}$$

Thus, all vectors of the form $p \otimes (q_1 \otimes r_1 + q_2 \otimes r_2)$ with $p, q_1, q_2, r_1, r_2 \in \mathbb{C}^2$, where (q_1, q_2) and (r_1, r_2) are linearly independent pairs, are equivalent to $e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$.

Equivalence Class 4:

$$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2 \tag{104}$$

$$(A \otimes B \otimes C) \cdot (e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2)) = Ae_1 \otimes Be_1 \otimes Ce_1 + Ae_2 \otimes Be_1 \otimes Ce_2 \tag{105}$$

Again, using the matrix forms above, we compute

$$Ae_1 \otimes Be_1 \otimes Ce_1 + Ae_2 \otimes Be_1 \otimes Ce_2 = \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} + \begin{bmatrix} a_2 \\ a_4 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} \otimes \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} \tag{106}$$

Thus, all vectors of the form $p_1 \otimes q \otimes r_1 + p_2 \otimes q \otimes r_2$ with $p_1, r_1, p_2, r_2, q \in \mathbb{C}^2$, where (p_1, p_2) and (r_1, r_2) are linearly independent pairs, are equivalent to $e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2$.

Equivalence Class 5:

$$(e_1 \otimes e_1 + e_2 \otimes e_2) \otimes e_1 \tag{107}$$

$$(A \otimes B \otimes C) \cdot (e_1 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_2) = Ae_1 \otimes Be_1 \otimes Ce_2 + Ae_2 \otimes Be_2 \otimes Ce_2 \tag{108}$$

Then this yields

$$\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} \otimes \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} + \begin{bmatrix} a_2 \\ a_4 \end{bmatrix} \otimes \begin{bmatrix} b_2 \\ b_4 \end{bmatrix} \otimes \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \left(\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} + \begin{bmatrix} a_2 \\ a_4 \end{bmatrix} \otimes \begin{bmatrix} b_2 \\ b_4 \end{bmatrix} \right) \otimes \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} \tag{109}$$

So, any vector expressible as $p_1 \otimes q_2 \otimes r_1 + p_2 \otimes q_1 \otimes r_1$, with $p_1, p_2,\ q_1, q_2$ linearly independent, is equivalent to $(e_1 \otimes e_2 + e_2 \otimes e_1) \otimes e_2$ under the action of $\mathrm{PGL}(2,2,2,\mathbb{C})$.

Equivalence Class 6:

$$e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \tag{110}$$

$$\begin{aligned} (A \otimes B \otimes C) \cdot (\quad & e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) = \\ & Ae_1 \otimes Be_1 \otimes Ce_2 + Ae_1 \otimes Be_2 \otimes Ce_1 + Ae_2 \otimes Be_1 \otimes Ce_1 \end{aligned} \tag{111}$$

Then this yields

$$\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} \otimes \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_2 \\ b_4 \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} + \begin{bmatrix} a_2 \\ a_4 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} \tag{112}$$

So, any vector expressible as $p_1 \otimes q_1 \otimes r_2 + p_1 \otimes q_2 \otimes r_1 + p_2 \otimes q_1 \otimes r_1$, with $p_1, p_2,\ q_1, q_2,\ r_1, r_2$ linearly independent, is equivalent to $e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$ under the action of $\mathrm{PGL}(2,2,2,\mathbb{C})$.

Equivalence Class 7:

$$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 \tag{113}$$

$$(A \otimes B \otimes C) \cdot (e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2) = Ae_1 \otimes Be_1 \otimes Ce_1 + Ae_2 \otimes Be_2 \otimes Ce_2 \tag{114}$$

Then

$$\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} + \begin{bmatrix} a_2 \\ a_4 \end{bmatrix} \otimes \begin{bmatrix} b_2 \\ b_4 \end{bmatrix} \otimes \begin{bmatrix} c_2 \\ c_4 \end{bmatrix}. \tag{115}$$

So, any vector expressible as $p_1 \otimes q_1 \otimes r_1 + p_2 \otimes q_2 \otimes r_2$, with $p_1, p_2,\ q_1, q_2,\ r_1, r_2$ linearly independent pairs, is equivalent to $e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$.

Now we will show that these seven equivalence classes are pairwise not equivalent under the PGL group.

Lemma 9. *The seven classes* eq. (95)-(113) *are pairwise not equivalent under the* PGL *group.*

Proof. Let $H_A \otimes H_B \otimes H_C = C^2 \otimes C^2 \otimes C^2$. We consider the Schmidt rank of Equivalence Class 2-7. If we consider the Schmidt rank of bipartite vectors in $H_A \otimes H_{BC}$, we know that:

Class 2, 3 has rank 1,

Classes 4, 5, 6, 7 have rank 2.

If we consider the Schmidt rank of bipartite vectors in $H_{AB} \otimes H_C$, we know that:

Class 2, 5 has rank 1,

Class 3, 4, 6, 7 have rank 2.

If we consider the Schmidt rank of bipartite vectors in $H_B \otimes H_{AC}$, we know that:

Class 2, 3, 4 has rank 1,

Class 5, 6, 7 has rank 2.

Thus, we only have to show that class 6 and 7 are not equivalent to each other.

For classes 6 and 7, we proceed as follows: If they are equivalent under PGL action, then there exist matrices P, Q, and R in $\mathrm{GL}(2, \mathbb{C})$ such that

$$(P \otimes Q \otimes R) \cdot (e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 \tag{116}$$

Suppose

$$P = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad Q = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \quad R = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}. \tag{117}$$

Then,

$$(P \otimes Q \otimes R) \cdot (e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) = \begin{bmatrix} a_1b_1c_2 \\ a_1b_3c_2 \\ a_3b_1c_2 \\ a_3b_3c_2 \\ a_1b_1c_4 \\ a_1b_3c_4 \\ a_3b_1c_4 \\ a_3b_3c_4 \end{bmatrix} + \begin{bmatrix} a_1b_2c_1 \\ a_1b_4c_1 \\ a_3b_2c_1 \\ a_3b_4c_1 \\ a_1b_2c_3 \\ a_1b_4c_3 \\ a_3b_2c_3 \\ a_3b_4c_3 \end{bmatrix} + \begin{bmatrix} a_2b_1c_1 \\ a_2b_3c_1 \\ a_4b_1c_1 \\ a_4b_3c_1 \\ a_2b_1c_3 \\ a_2b_3c_3 \\ a_4b_1c_3 \\ a_4b_3c_3 \end{bmatrix} = \begin{bmatrix} a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1 \\ a_1b_3c_2 + a_1b_4c_1 + a_2b_3c_1 \\ a_3b_1c_2 + a_3b_2c_1 + a_4b_1c_1 \\ a_3b_3c_2 + a_3b_4c_1 + a_4b_3c_1 \\ a_1b_1c_4 + a_1b_2c_3 + a_2b_1c_3 \\ a_1b_3c_4 + a_1b_4c_3 + a_2b_3c_3 \\ a_3b_1c_4 + a_3b_2c_3 + a_4b_1c_3 \\ a_3b_3c_4 + a_3b_4c_3 + a_4b_3c_3 \end{bmatrix}. \tag{118}$$

We require this to equal

$$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{119}$$

Thus, we obtain the system of equations

$$\begin{cases} a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1 = 1 & 1) \\ a_1b_3c_2 + a_1b_4c_1 + a_2b_3c_1 = 0 & 2) \\ a_3b_1c_2 + a_3b_2c_1 + a_4b_1c_1 = 0 & 3) \\ a_3b_3c_2 + a_3b_4c_1 + a_4b_3c_1 = 0 & 4) \\ a_1b_1c_4 + a_1b_2c_3 + a_2b_1c_3 = 0 & 5) \\ a_1b_3c_4 + a_1b_4c_3 + a_2b_3c_3 = 0 & 6) \\ a_3b_1c_4 + a_3b_2c_3 + a_4b_1c_3 = 0 & 7) \\ a_3b_3c_4 + a_3b_4c_3 + a_4b_3c_3 = 1 & 8) \end{cases} \tag{120}$$

We compare equation 2) and equation 3), then we get

$$\left((b_2 + b_4)c_1 + (b_1 + b_3)c_2\right)\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} + (b_3c_1 + b_1c_1)\begin{bmatrix} a_2 \\ a_4 \end{bmatrix} = 0. \tag{121}$$

Since $P \in \mathrm{GL}(2, \mathbb{C})$, we obtain $(b_2 + b_4)c_1 + (b_1 + b_3)c_2 = b_3c_1 + b_1c_1 = 0$.
Similarly, compare equation 2) and equation 4), then we get

$$2(b_3c_2 + b_4c_1)\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} + 2b_3c_1\begin{bmatrix} a_2 \\ a_4 \end{bmatrix} = 0. \tag{122}$$

Since $P \in \mathrm{GL}(2, \mathbb{C})$, we obtain $b_3c_2 + b_4c_1 = b_3c_1 = 0$.
Consider

$$\begin{cases} b_3c_1 = 0 \\ b_3c_1 + b_1c_1 = 0 \end{cases}, \tag{123}$$

We know that $b_1c_1 = b_3c_1 = 0$.
If $c_1 \neq 0$, then $b_1 = b_3 = 0$, contradicted with $Q \in \mathrm{GL}(2, \mathbb{C})$, so $c_1 = 0$.
We compare equation 6) and equation 7)
We have

$$\left((b_2 + b_4)c_3 + (b_1 + b_3)c_4\right)\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} + (b_1c_3 + b_3c_3)\begin{bmatrix} a_2 \\ a_4 \end{bmatrix} = 0. \tag{124}$$

Since $P \in \mathrm{GL}(2, \mathbb{C})$, we obtain $(b_2 + b_4)c_3 + (b_1 + b_3)c_4 = b_1c_3 + b_3c_3 = 0$.
Similarly, compare equation 5) and equation 7), then we get

$$2(b_1c_4 + b_2c_3)\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} + 2b_1c_3\begin{bmatrix} a_2 \\ a_4 \end{bmatrix} = 0. \tag{125}$$

Since $P \in \mathrm{GL}(2, \mathbb{C})$, we obtain $b_1c_4 + b_2c_3 = b_1c_3 = 0$.
Consider

$$\begin{cases} b_1c_3 = 0 \\ b_1c_3 + b_3c_3 = 0 \end{cases}, \tag{126}$$

We know that $b_1c_3 = b_3c_3 = 0$.

If $c_3 \neq 0$, then $b_1 = b_3 = 0$, contradicted with $Q \in \mathrm{GL}(2, \mathbb{C})$, so $c_3 = 0$. However, $c_1 = c_3 = 0$, $\det R = 0$, $R \notin \mathrm{GL}(2, \mathbb{C})$, contradicted!

5. Decomposition

Lemma 10. *Let $x \in \mathbb{C}^8$ with $\| x \| \leq 1$, and define the matrix $\rho = I_8 - xx^*$. Then ρ is positive semidefinite.*

Proof. For any vector $y \in \mathbb{C}^8$, consider the quadratic form

$$y^* \rho y = y^*(I_8 - xx^*)y = y^*y - y^*xx^*y = \| y \|^2 - |y^*x|^2. \tag{127}$$

By the Cauchy–Schwarz inequality, we have

$$|y^*x| \leq \| y \| \| x \|, \tag{128}$$

So

$$y^* \rho y \geq \| y \|^2 - \| y \|^2 \| x \|^2 = \| y \|^2 \, (1 - \| x \|^2). \tag{129}$$

Since $\| x \| \leq 1$, it follows that $1 - \| x \|^2 \geq 0$, hence $y^* \rho y \geq 0$ for all $y \in \mathbb{C}^8$. Therefore, ρ is positive semidefinite.

5.1. Positive semidefinite Decomposition

Now, we investigate for all $\rho = I_8 - xx^*$, where$|x| = 1$ whether there exists positive semidefinite Matrices A_i, B_i and C_i, such that

$$\rho = \sum_{i=1}^{n} A_i \otimes B_i \otimes C_i. \tag{130}$$

5.1.1. Equivalent Class 2

First, we examine equivalence class 2. Consider the simple case

$$x_0 = e_1 \otimes e_1 \otimes e_1 \tag{131}$$

It is easy to verify that

$$\begin{aligned} \rho = \quad & e_2 e_2^\top \otimes e_1 e_1^\top \otimes e_1 e_1^\top \\ + \quad & e_1 e_1^\top \otimes e_2 e_2^\top \otimes e_1 e_1^\top \\ + \quad & e_1 e_1^\top \otimes e_1 e_1^\top \otimes e_2 e_2^\top \\ + \quad & e_1 e_1^\top \otimes e_2 e_2^\top \otimes e_2 e_2^\top \\ + \quad & e_2 e_2^\top \otimes e_1 e_1^\top \otimes e_2 e_2^\top \\ + \quad & e_2 e_2^\top \otimes e_2 e_2^\top \otimes e_1 e_1^\top \end{aligned} \tag{132}$$

Now, let us consider the general case in this equivalence class

$$x_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \otimes \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{133}$$

Without loss of generality, we assume that each of the vectors

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{134}$$

Has magnitude 1 (we can scale each of them by a factor to make them all have magnitude 1). Then there exist unitary matrices P, Q, and R such that

$$(P \otimes Q \otimes R)x_1 = e_1 \otimes e_1 \otimes e_1. \tag{135}$$

We can consider $M = P \otimes Q \otimes R$.

On the one hand,

$$\begin{aligned} M^*\rho M &= M^*(I_8 - x_1x_1^*)M \\ &= M^*M - Mx_1(Mx_1)^* \\ &= I_8 - (Mx_1)(Mx_1)^* \\ &= I_8 - x_0x_0^*. \end{aligned} \tag{136}$$

On the other hand, we know that for a positive semidefinite matrix$S$ and a unitary matrix$T$, T^*ST is also a positive semidefinite matrix. So, all the new matrices in the decomposition are also positive semidefinite, we can conclude that for any vector in equivalent class 2, it can be decomposed in such a way.

5.2. Equivalent class 3, 4, 5

Lemma 11. *The positive semidefinite matrix X on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is separable iff $X^\Gamma \succcurlyeq 0$, where X^Γ is the image of X under partial transpose map.*

Lemma 12. *Under the PU group, any vector $v \in \mathbb{C}^m \otimes \mathbb{C}^n$ can be simplified to*

$$\textstyle\sum_{i=1}^{s} \lambda_i \, e_i^{(m)} \otimes e_i^{(n)}, \quad s \in N. \tag{137}$$

Proof. Let $V \in \mathbb{C}^{mn}$, and suppose we write

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{mn}o \end{bmatrix} \tag{138}$$

We can reshape V into a matrix $\mathcal{V} \in \mathbb{C}^{m \times n}$, such that:

$$\mathcal{V} = \begin{bmatrix} v_1 & v_{m+1} & \cdots & v_{m(n-1)+1} \\ v_2 & v_{m+2} & \cdots & v_{m(n-1)+2} \\ \vdots & \vdots & \ddots & \vdots \\ v_m & v_{2m} & \cdots & v_{mn} \end{bmatrix} \tag{139}$$

Suppose we take the singular value decomposition (SVD) of $\mathcal{V}$

$$\mathcal{V} = P\Sigma Q^*, \tag{140}$$

Where P and Q are unitary matrices, and Σ is a diagonal matrix with nonnegative entries. This can be written as

$$\mathcal{V} = \sigma_1 P_1 Q_1^* + \sigma_2 P_2 Q_2^* + \cdots + \sigma_k P_k Q_k^*, \tag{141}$$

Where $P_i \in \mathbb{C}^m$, $Q_i \in \mathbb{C}^n$, and $\sigma_i \geq 0$.
Define the extended orthonormal bases

$$P' := [P_1\ P_2\ \cdots\ P_m], \tag{142}$$

Where $P_1, \dots, P_m$ form an orthonormal basis of $\mathbb{C}^m$ and

$$Q' := [Q_1\ Q_2\ \cdots\ Q_n], \tag{143}$$

Where $Q_1, \dots, Q_n$ form an orthonormal basis of $\mathbb{C}^n$.
Now consider the transformation

$$(P')^{-1}\mathcal{V}(Q')^{-1}, \tag{144}$$

Which transforms $\mathcal{V}$ into the canonical form

$$\sigma_1 e_1^{(m)} e_1^{(n)*} + \cdots + \sigma_k e_k^{(m)} e_k^{(n)*}. \tag{145}$$

For a vector x in equivalence class 3, from Lemma 12, we know that it is equivalent to

$$x = e_0 \otimes (\lambda_0 e_0 \otimes e_0 + \lambda_1 e_1 \otimes e_1) \triangleq e_0 \otimes V, \tag{146}$$

Where $\lambda_0^2 + \lambda_1^2 = 1$ and $\lambda_0, \lambda_1 \geq 0$. We define the density matrix

$$\rho = I_8 - xx^* = I_2 \otimes I_4 - xx^*. \tag{147}$$

Expanding this expression, we get

$$\rho = (e_0 e_0^* + e_1 e_1^*) \otimes I_4 - (e_0 \otimes V)(e_0^* \otimes V^*) = e_0 e_0^* \otimes (I_4 - VV^*) + e_1 e_1^* \otimes I_4. \tag{148}$$

Hence, to prove that ρ is separable, it suffices to show that $I_4 - VV^*$ is separable. From Lemma 11, it is enough to prove that

$$(I_4 - VV^*)^\Gamma \succcurlyeq 0. \tag{149}$$

We compute

$$V = \begin{bmatrix} \lambda_0 \\ 0 \\ 0 \\ \lambda_1 \end{bmatrix}, \quad VV^* = \begin{bmatrix} \lambda_0^2 & 0 & 0 & \lambda_0\lambda_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_0\lambda_1 & 0 & 0 & \lambda_1^2 \end{bmatrix}, \tag{150}$$

So

$$I_4 - VV^* = \begin{bmatrix} 1-\lambda_0^2 & 0 & 0 & -\lambda_0\lambda_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\lambda_0\lambda_1 & 0 & 0 & 1-\lambda_1^2 \end{bmatrix}. \tag{151}$$

Applying partial transpose (on the second subsystem) we obtain

$$(I_4 - VV^*)^\Gamma = \begin{bmatrix} 1-\lambda_0^2 & 0 & 0 & 0 \\ 0 & 1 & \lambda_0\lambda_1 & 0 \\ 0 & \lambda_0\lambda_1 & 1 & 0 \\ 0 & 0 & 0 & 1-\lambda_1^2 \end{bmatrix}. \tag{152}$$

We now verify that $(I_4 - VV^*)^\Gamma \succcurlyeq 0$. It suffices to show the 2×2 block

$$\begin{bmatrix} 1 & \lambda_0\lambda_1 \\ \lambda_0\lambda_1 & 1 \end{bmatrix} \succcurlyeq 0. \tag{153}$$

From Lemma 2, this matrix is positive semidefinite if and only if

$$1 - (\lambda_0\lambda_1)^2 = (1-\lambda_0\lambda_1)(1+\lambda_0\lambda_1) \geq 0. \tag{154}$$

Since $\lambda_0, \lambda_1 \geq 0$, clearly $1 + \lambda_0\lambda_1 \geq 0$. Also, by the AM–GM inequality

$$\lambda_0\lambda_1 \leq \frac{\lambda_0^2+\lambda_1^2}{2} = \frac{1}{2}, \tag{155}$$

So $1 - \lambda_0\lambda_1 \geq \frac{1}{2} > 0$. Therefore, the entire expression is non-negative, and $(I_4 - VV^*)^\Gamma \succcurlyeq 0$, proving that $I_4 - VV^*$ is separable. Thus, the matrix equivalence class 3 is separable. For classes 4 and 5, note that they are simply permutations of the three vector components in class 3. Since separability is preserved under local unitary operations and permutations, these classes are also separable.

5.3. Equivalent class 6

For vectors x in **equivalence class 6**, consider

$$x = (W \otimes Y \otimes Z)(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1), \tag{156}$$

Where $W, Y, Z \in \mathrm{GL}(2, \mathbb{C})$. We define

$$\rho = I_8 - xx^*. \tag{157}$$

We may find a unitary matrix

$$G \in \mathrm{U}(2, \mathbb{C}) \otimes \mathrm{U}(2, \mathbb{C}) \otimes \mathrm{U}(2, \mathbb{C}), \tag{158}$$

Such that

$$G(W \otimes Y \otimes Z) = \begin{bmatrix} w_1 & w_2 \\ 0 & w_3 \end{bmatrix} \otimes \begin{bmatrix} y_1 & y_2 \\ 0 & y_3 \end{bmatrix} \otimes \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix}. \tag{159}$$

Then we consider

$$G\rho G^* = I_8 - Gx(Gx)^*. \tag{160}$$

Under this transformation,

$$Gx = \left(\begin{bmatrix} w_1 & w_2 \\ 0 & w_3 \end{bmatrix} \otimes \begin{bmatrix} y_1 & y_2 \\ 0 & y_3 \end{bmatrix} \otimes \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix} \right) (e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1). \tag{161}$$

This yields a vector of the form

$$Gx = a\, e_1 \otimes e_1 \otimes e_2 + b\, e_1 \otimes e_2 \otimes e_1 + c\, e_2 \otimes e_1 \otimes e_1 + d\, e_1 \otimes e_1 \otimes e_1, \tag{162}$$

Where $a, b, c > 0,\ d \geq 0$, and $a^2 + b^2 + c^2 + d^2 = 1$.

Thus, we transform the problem to discussing the separability of $\rho = I_8 - yy^*$, where

$$y = a\, e_1 \otimes e_1 \otimes e_2 + b\, e_1 \otimes e_2 \otimes e_1 + c\, e_2 \otimes e_1 \otimes e_1 + d\, e_1 \otimes e_1 \otimes e_1, \tag{163}$$

Where $a, b, c > 0,\ d \geq 0$, and $a^2 + b^2 + c^2 + d^2 = 1$.

5.4. Equivalent class 7

Similarly, for vectors x in **equivalence class 7**, consider

$$x = (W \otimes Y \otimes Z)(e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2), \tag{164}$$

Where $W, Y, Z \in \mathrm{GL}(2, \mathbb{C})$. We may find a unitary matrix

$$G \in \mathrm{U}(2, \mathbb{C}) \otimes \mathrm{U}(2, \mathbb{C}) \otimes \mathrm{U}(2, \mathbb{C}), \tag{165}$$

Such that

$$G(W \otimes Y \otimes Z) = \begin{bmatrix} w_1 & w_2 \\ 0 & w_3 \end{bmatrix} \otimes \begin{bmatrix} y_1 & y_2 \\ 0 & y_3 \end{bmatrix} \otimes \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix}. \tag{166}$$

Then we consider

$$G\rho G^* = I_8 - Gx(Gx)^*. \tag{167}$$

Under this transformation,

$$Gx = \left(\begin{bmatrix} w_1 & w_2 \\ 0 & w_3 \end{bmatrix} \otimes \begin{bmatrix} y_1 & y_2 \\ 0 & y_3 \end{bmatrix} \otimes \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix} \right) (e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2). \tag{168}$$

This yields a vector of the form

$$Gx = a\, e_1 \otimes e_1 \otimes e_1 + p \otimes q \otimes r \tag{169}$$

Where $a > 0$, and $p, q, r \in \mathbb{C}^2$. Thus, we transform the problem into discussing the separability of $\rho = I_8 - xx^*$, where

$$Gx = a\, e_1 \otimes e_1 \otimes e_1 + p \otimes q \otimes r \tag{170}$$

Where $a < 0$, and $p, q, r \in \mathbb{R}^2$ and the components of the vector p, q, r are non-negative and real.

6. Conclusion

We have investigated the absolute separability problem for the case of 8×8 positive semidefinite matrices. We have classified the vectors in a finitely-dimensional bipartite Hilbert space and constructed their equivalence classes in terms of the so-called Schmidt rank, under both PGL and PU groups. Then we have found seven equivalence classes of x under the PGL group, using which we have partially proved that the conjecture that the positive semidefinite matrix $I_8 - xx^*$ with an eight-dimensional unit vector x is absolutely separable. Our results provide a foundation for complete proof of the Conjecture in future work. An interesting problem arising from this paper is to extend the conjecture to the case of a four-partite Hilbert space, that is, whether $I_{16} - yy^*$ is absolutely separable with any 16-dimensional unit vector y.

Acknowledgements

During the completion of this project, I have received tremendous, valuable support and assistance from many teachers and classmates. I would like to extend my sincere gratitude to all those who have offered me care and help. During my studies, I also benefited greatly from the support of my classmates and friends. I am grateful for their help and companionship in my daily academic life. Once again, I wish to express my sincere appreciation to everyone who has supported and encouraged me.

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