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Escape Orbits in the Three-Body Problem

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Abstract. The escape orbit in the three-body system is a crucial topic in the field of astrophysics, helping people to research the chaotic system, stimulate the evolution of some complex celestial systems, and design the spacecraft trajectory. Extensive research has been conducted on this topic, but there are still some limitations. This paper focuses on reviewing and analyzing escape orbits in the problem of three bodies. Specifically, this paper explains the basic mechanics conditions and analyzes findings from the general and restricted three-body problems (CRTBP and ERTBP). Key tools such as Poincaré sections, ergodic hypothesis, and fractal basin boundaries are introduced to explain escape mechanisms. The study also highlights recent progress in the escape of a celestial body: spacecraft escape orbit design, relativistic effects on escape, and statistical analysis. The review of gravity-assist and low-energy escape orbit design in this paper provides theoretical support for space missions. Future research should include more comprehensive models, initial conditions, and higher-order physical effects.

Keywords: Spacecraft; escape orbits; celestial body; restricted three-body problem.

1. Introduction

In astrophysics, the question of three bodies is always a fundamental issue. Understanding the escaping orbit in the three-body problem provides a simpler and more accurate way for people to research the chaotic system and stimulate the evolution of some complex celestial systems in the universe. It can also be used to research ballistic capture and gravity assists to generate breakthroughs in aerospace technology [1].

Researchers have conducted extensive research on escape behavior. In the late 20th century, Aarseth et al., who assumed that the three celestial bodies have equal mass and zero initial velocities, studied 5000 close triple approaches [2]. They discovered that when the escaping celestial body passes close to the center of mass of the other two bodies in an almost straight line and has a direction of motion that is nearly perpendicular relative to the line joining the other two bodies, leaving a near triple-body approach in a fly-by maneuver increases the likelihood of escape [2]. The escaping velocity is relatively large in this situation [2]. The escape behavior of a small celestial body under the gravitational attraction of two considerably heavier things that travel in separate circles around their shared center of gravity in the same plane and have the same Kepler frequency was studied by Jan Nagler in 2004 [3]. The researcher found that when the mass ratio between the two heavy bodies is significantly different such as earth-moon system (mass ratio is 1/82.3), the probability is extremely small for the small body with the initial position near the larger body to escape from the system, and the small body tends to follow bounded motion or collide with the large body [3]. In 2019, Nicholas based on the fact that the bound, non-hierarchical three-body system (masses of the three celestial bodies are similar and their distances between each other are not significantly different) tends to evolve into a single escaping star and a stable bound binary [4]. They derived the overall probability distribution of energy, binary eccentricity, and orientation of the remaining two-body system in three-dimensional space using the ergodicity hypothesis, which is very accurate [4].

The above papers deepen people's comprehension of escape dynamics within systems involving three interacting celestial bodies. Building upon previous findings, this paper aims to analyze the achievements in the research of escape paths in the context of the three-body problem, explain the

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fundamental concepts behind escape orbits, and investigate the applications to aerospace technology. The paper will also point out the unsolved issues in the three-body problem, providing directions for future research.

2. Theoretical Basis

2.1. Fundamental Knowledge about the Three-Body Problem

Behaviors of a group of three celestial bodies are studied as they are affected by each other's gravitational forces. Compared to the two-body case, such systems are extremely sensitive to their initial settings, and there is no common law to describe the movement of these three bodies. It was first proposed by Newton, and Euler studied a special case where three celestial bodies maintain a collinear configuration and move at the same angular velocity around a common center of mass under the influence of gravity. However, Euler's solution is difficult to persist in real celestial systems due to the extreme sensitivity to initial conditions and system's rapid departure from collinearity. After that, Lagrange's solution is proposed to describe the dynamic behavior of three astronomical bodies that maintain an equilateral triangle configuration and rotate uniformly around the central mass point of the system.

Compared with Euler's solution, Lagrange's solution exhibits good stability under specific mass ratio conditions. In the restricted three-body problem, the Lagrangian system organizes five special points (L1-L5) in a system composed of two massive celestial bodies (such as the Sun and Earth), at which small celestial bodies can maintain a fixed position between the two celestial bodies in a rotating reference frame. The L1 point is located on a line connecting two main celestial bodies (such as the Sun and the Earth), close to the smaller mass celestial body (such as the Earth), so the Sun, L1, and the Earth are arranged in a straight line. L2 is located on the side of a lower mass celestial body (such as Earth) away from a higher mass celestial body, so the Sun, Earth, and L2 are collinear. Then, Poincaré conducted deep research on it in the 19th century, revealing for the first time the existence of chaotic motion in nonlinear systems. In recent years, researchers have begun to extensively use numerical methods to calculate escape rates, resonance structures, and fractal boundaries. Although the result is not an analytical solution, these studies reveal the macroscopic laws of the orbit of the three-body system.

2.2. Introduction of the Escaping Orbit

2.2.1. Basic mechanics.

The reasons for a celestial body to escape from a three-body system are mainly due to one of the following mechanisms.

The celestial body has sufficient energy to break free from the gravitational binding when the three-body system's total energy is positive. The three-body system must split, resulting in an escaping body and a binary system [5]. Scramble refers to a period during which three stars do not have a fixed paired structure and move in mixed motion with each other. When more scrambles happen in the system, the celestial body is more likely to escape due to the chaotic motion [4]. The resonance means periods of orbits of celestial bodies are in integer ratios, which will let the influence gradually accumulate, many if not most, making escape easier.

2.2.2. Escape conditions.

The escape of the celestial body needs to meet following conditions (Some specific numbers may vary depending on different experiments): 1) The distance between it and the system's mass center is at least 5 parsec away; 2) It is moving away from the center of mass of the system; 3) It's energy must be positive; 4) It achieves above three conditions for at least 100 system's natural timescale [6]. Escape rate refers to the proportion of orbits that a celestial body escapes from the three-body system within a given time range, out of the total number of orbits under a group of starting conditions. This rate measures the probability of celestial escape in a particular region or under specific initial conditions, and is an important physical quantity for describing system stability and chaotic characteristics.

2.3. The General Three-Body Problem and the Restricted Three-Body Problem

2.3.1. The general three-body problem.

Lower total angular momentum within the system, combined with a more chaotic evolution, tends to raise the likelihood of producing an escaping object and a binary configuration, and the binary system tends to have higher eccentricity [4]. Additionally, the orbital angular momentum typically points in the same direction as that of the original system [4].

2.3.2. The restricted three-body problem.

The key assumption holds that the third celestial object has a negligible mass compared to the other two, exerting no significant gravitational influence on their motion. The only gravitational effect is from two heavier celestial bodies on the body with a small mass. The limited problem of three celestial bodies has two key models: the circular and elliptical forms, known as the CRTBP and ERTBP, respectively. In CRTBP, two heavy celestial bodies rotate around a common center of mass in a circular orbit, while the mass of the small mass body can be ignored, and its motion is limited to the plane of the main star's orbit. In the sun-earth-spacecraft system, Lagrangian points L_2 and L_1 can be important starting points for a spacecraft to escape, but L_2 can better reduce the energy consumption of a spacecraft and adapt to more missions about exploring the outer space than L_1 [7]. In ERTBP, two heavy celestial bodies rotate around a common center of mass in an elliptical orbit, while the mass of the smallest body can be ignored. ERTBP is closer to the three-body system in reality than CRTBP. The eccentricity is a crucial factor in ERTBP. When the eccentricity is relatively low, the system is more stable, and satellite-like motion around one heavy body and planetary motion surrounding the two heavy bodies in a binary system are more likely to happen. However, when the eccentricity is relatively high, satellite-like motion around one heavy body will still exist for $e < 0.9$, and for $e > 0.8$, the planet-like motion surrounding the binary system of two heavy bodies will vanish entirely. When $e > 0.9$, the system is very unstable, and the small celestial body is more likely to escape or collide with one body [8].

2.4. Related Algorithms and Models

2.4.1. Poincaré section.

Poincaré Section is a tool that selects a cross-section in multidimensional phase space to record the state of the track passing through this section in order to simplify complex systems and facilitate the analysis of certain characteristics of the system. For example, Jan Nagler used Poincaré sections with bounded motion, escape orbit, and collision having different colors, and the distribution of cross-sectional intersection points can intuitively reflect the motion characteristics of small celestial bodies in restricted three-body systems [3].

2.4.2. Ergodic hypothesis.

Assuming that non-hierarchical and similar-mass triples will uniformly explore the phase-space volume that is accessible to them, the ergodic hypothesis allows researchers to derive the likelihood distribution of possible outcomes [4]. The three-body problem is sensitive to initial conditions, but this method can solve the problem by transforming it into statistical results [4].

2.4.3. Fractal basin boundaries.

Fractal basin boundaries are a geometric feature of the chaotic system within the three-body dynamics, which means that minor differences in initial conditions may lead to different outcomes (escape, collision, or bondage). The geometry of the boundaries that contain the mixture of initial conditions is nearly fractal, and the prediction difficulty of a three-body system can be qualified through fractal boundaries and fractal dimension [8].

3. Escape of a Celestial Body

3.1. The Design of Escape Orbits for a Spacecraft

In 2021, Andrea developed a new and improved model of escape trajectories based on a patched three-body model (a trajectory modeling method that connects multiple restrictive three- body systems in sequence at specific patching surfaces to handle each three-body problem in each segment and simplify the design and analysis of spacecraft trajectories in complex three-body systems) to transfer a spacecraft effectively Starting within the Earth-Moon system and extending to regions beyond it [9]. They divided the escape problem into two parts: the internal problem from the departure point to the switching surface and the external problem from the switching surface to the target interception [9]. They constructed the internal trajectory set and the external trajectory set [9]. Then one trajectory is selected from each set, and the researcher compares the difference between velocity magnitude, position vectors, the angle between position vectors, and the angle between velocity vectors on the switching surface to see if it is in the allowable range of error [9]. When they finished selecting two trajectories, they used methods such as multiple shooting to improve the design of the two trajectories and get the final escape orbit [9]. Their strategy significantly simplifies the escape orbit design process in the three-body power system, not only improving the decalculation efficiency, but also enhancing the orbit adaptability, and providing a practical solution for future long-earth transfer missions of manned or exploration spacecraft [9].

3.2. The Impact of Relativistic Effects on Escape

In 2021, Boekholt et al. researched the impact of relativistic effects on escape in the Pythagorean three-body problem [6]. They found something interesting: the dynamic behavior of the system is approximately consistent with Newtonian mechanics, and the three body interaction usually ends with a stable binary star and an escaping celestial body when the combined mass of the system $\leq 10M_{\odot}$; when that increases to $10^7 \mathrm{M}_{\odot}$, relativistic effects through gravitational wave emission become more dominant, but most of the system still result a stable binary star and an escaping celestial body; when that is between $10^7 \mathrm{M}_{\odot}$ and $10^9 \mathrm{M}_{\odot}$, all systems end in a merger; for that $\geq 10^9 \mathrm{M}_{\odot}$, in the first close approach, all systems lead to a merger of gravitational waves. This study not only has a positive impact on escape dynamics in the three-body system, but also contributes to research on supermassive black holes [6].

3.3. Statistical Analysis of a Binary System and an Escape Body

In 2021, a statistical prediction method was proposed for the escape problem of three body systems by Barak Kol, and he replaced the traditional method of phase-space volume with phase- -space flux as the key factor in probability calculation and reconstructed the calculation framework of probability given the additional variables (the regularized phase space volume and the chaotic absorptivity). [10]. The predicted escape probability was more consistent with the actual simulation results compared to results from previous methods [10]. He concluded that the larger the maximum angular momentum at a given energy and mass that a binary system can have, the easier it is for the system to break into a binary system and an escape body [10].

$$k_s\text{=}(m_a m_b)(\frac{\mathrm{G}m_a m_b)^2}{m_a+m_b},\ \mathrm{l_s} = \sqrt{\frac{\mathrm{k_s}}{(-2\epsilon)}}, Ps \propto k_s^3 \quad (1)$$

ϵ: the given energy of the binary system. m_a, m_b: mass of two bodies in the binary system Ps: the probability to escape. l_s: the maximum angular momentum of the binary system for given ϵ. k_s: the binary constant. G: Newton's gravitational constant.

4. Applications

4.1. The Gravity-Assist

The gravity-assist maneuver involves a three-body configuration consisting of the spacecraft, the assisting body, and the central celestial body around which the other two orbit [11]. A close encounter with a moving celestial body allows the spacecraft to reshape its orbital path, saving a significant amount of fuel and travel time [11]. For this reason, it was utilized in several interplanetary activities, including Galileo, Mariner, and Voyager [11].

4.2. Low-Energy Escape Orbit Design

In a three-body system, due to the complexity of the system, small celestial bodies can escape while the total energy remains negative or slightly positive, which is called "low-energy escape". For example, Villac explored low-energy escape trajectory design based on the Hill-type three-body approximation, which represents a special form within the circular restricted three-body system. Within this framework, a coordinate system is placed at the center of a primary body (e.g., a minor celestial object or a satellite revolving around a planet), and the ratio of masses between the primary and secondary bodies is assumed to approach zero, while adjusting the position values to keep them finite. They found that when trajectories with a Jacobi constant (a conserved quantity in a rotating reference frame, combining the kinetic energy, effective potential energy, and centrifugal potential of a small celestial body) just above the critical value of $J_{L_{1,2}}$ (the smallest J for a body to escape at L_1 or L_2), zero velocity surfaces (a surface where a body has a velocity of 0, only moves within the wrapped area, and doesn't go out of the surface) will open escape locations at L1 and L2, corresponding to the lowest energy possible for escaping trajectories [12]. r is the separation between the central body and the small object, and orbital speed increases with the decrease in r [12]. Periapsis has the smallest r, which means the spacecraft has the highest kinetic energy and the lowest potential energy [12]. A tangential burn at periapsis achieves the minimum ΔV to escape along an escaping trajectory, saving a significant amount of fuel [12].

5. Limitations and Future Directions

The three-body problem is always a research challenge in astronomical science and aerospace dynamics due to its chaos and complex orbital structure. Although the research about the escape orbit in the three-body system has already made significant achievements, there are still many drawbacks, which limit its application in simulating complex celestial system modeling and space mission orbit design. Firstly, many researchers still focus on the redistricted three-body problem (the circular restricted three-body problem and the elliptical restricted three-body problem). It simplifies the model excessively, and it can only be applied to some space scenarios and experimental simulations; it does not contribute to more complex space scenarios and theoretical research. Secondly, many scientists use statistical solutions for the non-restrictive three-body problem. However, their findings rely upon an idealized framework with simplified starting parameters. For example, they only focus on the structure of binary star systems and an escape star, and assume that the total energy is negative [13]. In complex situations, this prediction may not be applicable. Thirdly, many studies are still limited to the framework of Newtonian gravity and have not considered important physical mechanisms such as general relativity effects and tidal dissipation. In the future, research on three-body escape orbits should break through existing limitations from multiple perspectives: firstly, pay more attention to the theoretical development of no-restricted three-body and multi-body problems; second is to introduce a wider range of initial parameters, more complex models, and higher-order physical mechanisms in the research.

6. Conclusion

This paper reviews the fundamental knowledge related to research and applications on escaping trajectories within a three-body system. The paper begins with mechanics and conditions for escaping. Then, the paper explains escaping in different models for the three-body system (restricted and non-restricted three-body problems), highlighting key tools like Poincaré section, ergodic hypothesis, and fractal basin boundaries. By analyzing scientists' research on escape problems, theoretical achievements have been more comprehensively summarized in this paper. In terms of application, the review of gravity-assist and low-energy escape orbit design provides theoretical support for space missions. Studying the problem of the escape orbit in the three-body problem can deepen people's exploration of the universe and better simulate the motion of celestial bodies, and it is crucial in spacecraft trajectory planning across the cosmos. However, current researchers still have limitations: many researchers idealized the conditions and models; many researchers ignored some important mechanics, for example, general relativity effects. In the future, researchers should study these issues more seriously.

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