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Applications of Lagrange Points in Circular Restricted 3-Body Problem

Junhao Yang

Aiglon College, 1885, Vaud, Switzerland

yanjun23@aiglon.ch

Abstract. The Circular Restricted Three-Body Problem (CR3BP) is a simplification of the general three-body problem with essential assumptions: two massive primaries and an infinitesimal third body influenced solely by the gravity of the primaries. The equilibrium points that emerge from the resulting simplified system—Lagrange points—have become crucial locations for space missions due to their special stability characteristics. This article examines the usefulness of these points, specifically L1, L2, L4, and L5, for space exploration and fundamental physics experiments. It discusses the prospects of using these points for gravitational wave detection, gravitomagnetic field measurement, and relativistic time delay observations. The implications of Lagrange points on satellite deployment, asteroid concentration, and long-duration space missions are also covered. These stable points offer significant opportunities for a number of observational and scientific endeavors, enabling both Earth-bound and deep-space exploration. The study of Lagrange points continues to bridge theoretical celestial mechanics with practical applications in modern astrophysics and space technology.

Keywords: Lagrange Points; Circular restricted 3-Body Problem; Application.

1. Introduction

The three-body problem of space physics refers to the way in which three objects behave as they attract one another through gravity. The two-body problem has a known solution, but the three-body problem behaves unpredictably and has no general solutions. This makes it extremely difficult to obtain solutions for any initial positions. It's difficult because slight movements at the beginning can result in large movements, leading to haphazard paths and circular motion. Circular Restricted Three-Body Problem (CR3BP) makes some assumptions: there are two big objects orbiting in a circle around a common center, and there's a third, extremely small object. The third object experiences the force of the big objects but does not alter the way they're moving. It prepares us to learn about Lagrange points, five locations where the force of the two big objects cancels out the force that would attempt to pull the third object off course. The assumptions of the CR3BP leads to a both numerical and analytical solution, making the problem less challenging than the general three-body problem, and assist in the planning of spacecraft missions and physics experiments.

2. Presenting the Lagrange Points in Respect of their Configuration for a Sun-Earth or Earth-Moon System

The Circular Restricted Three-Body Problem (CR3BP) is a simplification of the general Three-Body Problem. In this case, two bodies, called the primaries, are very massive and move under each other's gravity. The third body has negligible mass, so small that it does not affect the motion of the two primaries at all. However, the third body is influenced by the gravitational pull from both primaries.

These restrictive assumptions can represent several physical situation in the universe like Earth-Moon or Earth-Sun system with a probe, satellite, and etc. In such case, constant-pattern solution of CR3BP, including all Lagrange points (L_1, L_2, L_3, L_4, L_5) are considered important. Lagrange points, named after Joseph-Louis Lagrange, the man who discovered them, are points with equilibrium where all forces acting on it are cancelled out. In Earth-Sun system, there are two primaries: Sun with mass m_1 and Earth m_2, where the mass ratio μ is always less than one

$$\frac{m_2}{m_1} = \mu \tag{1}$$

Except in the 'Copenhagen problem', when it is set equal to one [1]. This forms a circular orbit with a common focus close enough to m_1. In this case, the gravitational force exerted by the Earth on the Sun is considered negligible, resulting in the sun being stationary and orbited by the Earth. A simple analysis to the third body m_3 is that it is affected by the gravitational attraction from m_1 and m_2, two forces F_1 (pointing toward m_1) and F_2 (pointing toward m_2) with opposite direction; However, it does not interact with the circular orbital motion of m_1 and m_2. The distance between bodies m_1 and m_2 is denoted as r and the distance between bodies m_1 and m_3 is denoted as x (shown in Fig. 1)

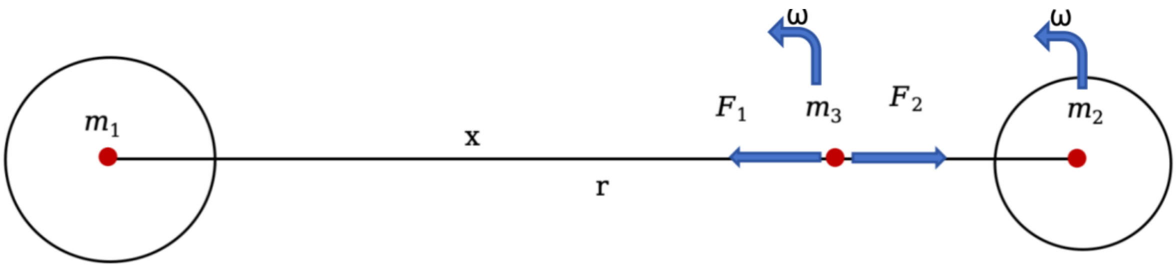

Figure 1. Not to scale (picture credit: original)

According to Newton's Universal Law of Gravitation, the gravitational force between any two bodies can be written as

$$F = G\frac{m_1 m_2}{r^2} \tag{2}$$

Where G is the Gravitatonal constant. Hence, the equation of two balanced force, $F_1 = F_2$, can be wriiten as

$$G\frac{m_1 m_3}{x^2} = G\frac{m_3 m_2}{(r-x)^2} \tag{3}$$

Simplified into

$$\left(\frac{r-x}{x}\right)^2 = \frac{m_2}{m_1} \tag{4}$$

Where the distance x between point L_1 and m_1 is found as

$$x = \frac{r}{1+\sqrt{\frac{m_2}{m_1}}}, (x < r) \tag{5}$$

In real planatary movement, celestial bodies like the Earth is considered orbiting with m_3, sharing a same constant angular speed ω in a inertial frame. Since the restricted Three-Body problem can be easily understood in a rotating reference frame, the third body m_3 in the coordinate system also shares the angular speed ω.

If it neglects the third body, the two bodies circular motion suggests a pair of balanced force, Gravitational force and Centripetal force, denoted as

$$F_g = F_c \quad (6)$$

Both forces can be written in terms of m, ω, G and r

$$\frac{m_1 m_2}{r^2} = m_2 \omega^2 r \quad (7)$$

$$\omega^2 = \frac{G m_1}{r^3} \quad (8)$$

For the third body's equation of $\sum F = F_c$ can be written as

$$F_1 - F_2 = F_c \quad (9)$$

Written in terms of m, ω, G and r

$$G\frac{m_1 m_3}{x^2} - G\frac{m_3 m_2}{(r-x)^2} = m_0 \omega^2 x \quad (10)$$

Replacing ω^2 since $\omega^2 = \frac{G m_1}{r^3}$ (7)

$$m_1(r-x)^2 - m_2 x^2 = \frac{m_1}{r^3} x^3 (r-x)^2 \quad (11)$$

For this equation the roots of fifth-order polynomialst can be found by using Newton Raphson.

By analysing the problem with vectors and cosidering the forces acting on the small mass are in equilibrium, it finds these points by setting the gradient of the potential ∇U(x, y) to zero. This condition balances the gravitational and centrifugal forces. It introduces $\alpha = \frac{m_1}{m_1+m_2}$. An analytic solution can easily be found when α is small, giving the location for L_1, L_2, L_3, L_4, L_5 [2]

$$L_1 = R(1 - (\frac{\alpha}{3})^{\frac{1}{3}})$$
$$L_2 = R(1 + (\frac{\alpha}{3})^{\frac{1}{3}})$$
$$L_3 = -R(+(\frac{5\alpha}{12}))$$
$$L_4 = (\frac{R(m_1 - m_2)}{2(m_1 + m_2)}, \frac{\sqrt{3}}{2} R)$$
$$L_5 = (\frac{R(m_1 - m_2)}{2(m_1 + m_2)}, -\frac{\sqrt{3}}{2} R \quad (12)$$

In which, L_1, L_2, L_3 are collinear with the other two bodies with L1 located within m1 and m2; L2 outside M1 and M2 on the right-hand side; and L3 on the opposite side of m2. L_4, L_5 at the top of the two symmetrical equilateral triangles that the other two bodies from with itself, shown by Fig. 2 and Fig. 3.

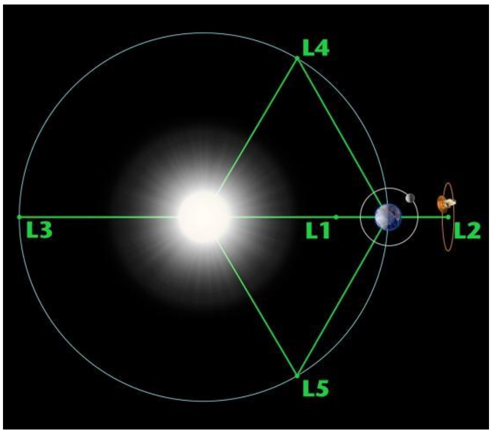

Figure 2. Geometric illustration of L1-L5 [3]

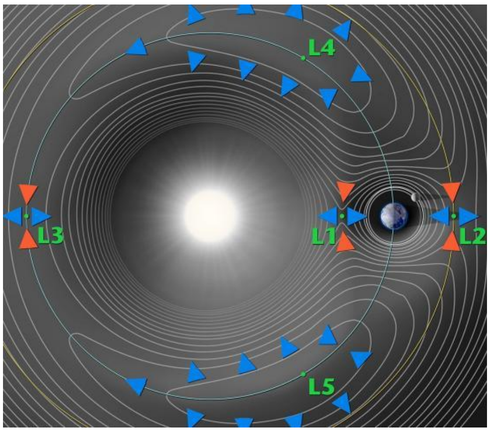

Figure 3. Gravitational illustration of L1-L5 [3]

3. Application of Lagrange Points

Scientists have found numerous advantages of deploying probes to near Lagrange points. Saddle points shown in Fig.3 (L1, L2 and L3) are considered unstable compare to the stable points L4 and L5 that set on the apex of the equilateral triangle, which depends on the ratio of the two primaries' masses (unstable if $0.03852 < \mu < 0.96148$) [4]. Objects placed at meta-unstable point L1, L2 and L3 can experience additional forces caused by the gravitational field of other celestial bodies like the Moon. Therefore, the object will orbit in a small path in a region instead of staying in rest, making most space requiring instruments like thrusters for adjustment for station-keeping with dynamical instability [5]. However, the fuel-consumption of the spacecraft hasn't been an unapproachable problem in real space mission. The optimal trajectory at Lagrange point can be formulated as an optimal control problem, which can be further analyzed with computer simulation [6].

For L1 in particular, this provides an advantage for observation of satellite remote sensing. During Low-Earth Orbit (LEO), satellites are placed at an altitude of 500km-2000km, providing a scan of small portion of the surface at a given time with high resolution. During Geostationary Orbits (GEO), satellites are placed at an altitude of 36,000km above the Earth, providing a continues observation of a large portion of the planet with relatively lower resolution. Since L1 is located on the far side between the Moon and the Earth ($M_{Earth} > M_{Moon}$) in an Earth-Moon system, the ideal distance of, 1.5 million km from the Earth, makes it great enough for an observation of an entire disk of the Earth at once with single platfrom. In fact, the placement of satellites at L1 can be tracked to ealy era of space exploration. The iconic "Blue Marble" image shown in Fig. 4, taken by the Apollo 17 crew in 1972 while they were traveling to the Moon, sparked public interest in Earth observation.

Figure 4. High-resolution photographs of Earth [7]

This spectacular 'blue marble' image is the most detailed true-color image of the entire Earth to date. Using a collection of satellite-based observations, scientists and visualizers stitched together months of observations [7].

Even in other situations L1 has always been an ideal position of observation, e.g. in the Earth-Sun system. The launch of the satellite SOHO in 1995 also involves L1. This project features in Solar Wind Plasma Experiment (Measures the properties of solar wind) and Michelson Doppler Imager (Studies solar oscillations and provides insight into the Sun's internal structure) [8]. To reiterate, L1uninterrupted view of the entire disk of Earth which allows scientists to gather data on climate, weather patterns, solar radiation, and other global phenomena. Both L1 and L2 are a special location for DSCOVR (Deep Space Climate Observatory), positioning on the Ecliptic plane where the net gravitational pull of the Earth and Sun equals the centripetal force required to orbit the Sun with the same orbital angular velocity as the Earth. To stable the satellites placed in L1 and L2, anon-repeating elliptical Lissajous orbit must be implemented, in order to deal with interference from solar flux. This orbit is perpendicular to the Sun-Earth plane, forming a tilt of 4°–12° (or 2°–12° after March 2020). The angle of the deviation from the Sun-Earth plane is noted as Azimuth Angle (ϕD), shown in Fig. 5. This angle could be essential to the observation at the satellite, as the entire disk of the Earth tends to be observed completely when ϕD at 180 degrees (yet some there will be single digit percentage loss due to Lissajous orbit) [9].

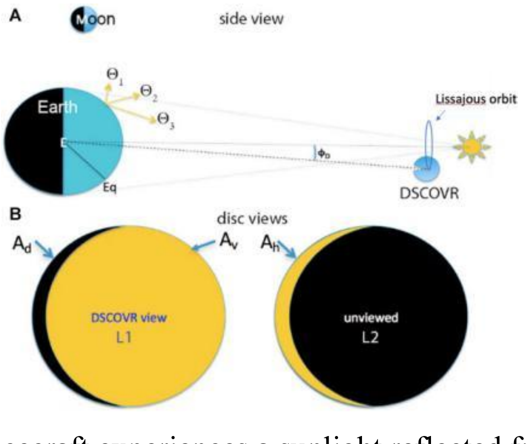

Figure 5. Spacecraft experiences a sunlight reflected from Earth [9]

The spacecraft also experiences a sunlight reflected from Earth, forming an angle denoted by the letter θ shown in Fig.5. The multi-dimensional sunlight scatters in various directions and a larger value of θ means higher rate of diffuse, requiring scientists to considered this variation. In addition, L1 provides continuous observation of the daylight side of Earth because it is positioned between the Earth and the Sun (shown in the Fig.5 as the golden area Av). However, the golden area Ah on the right side represents the sunlit area not seen by DSCOVR. Spacecraft at L1 can observe the sunlit side of Earth continuously, making it an ideal position for monitoring. However, L1 does not see the night side of Earth (the side that is in the shadow of the Earth), which limits the ability to observe Earth during the night. L2, on the other hand, is positioned on the opposite side of Earth from the Sun, even with inconsistent observation. Its information of the nighttime side of Earth, allowing for the study of phenomena like infrared radiation, which is emitted from Earth during the night. The infrared observations at L2 allow the study of temperature variations on Earth, atmospheric composition, and other nighttime weather phenomena that L1 cannot observe. Together, L1 and L2 allow for complete global coverage of Earth's surface, both during day and night, which is crucial for understanding the Earth's climate, weather systems, and solar influence on our planet.

Similarly, L5 is an optimal place for space weather research, featuring by an operational CME propagation model. L5's special 60-degree trailing position gives 4-5 days warning for solar events, that allows full tracking of earth [10]. Tracking CMEs (Coronal Mass Ejection), a solar event, continuously within the optimal Thomson scattering surface can reduce current arrival time uncertainties and speed errors. On the other hand, unlike all the collinear point L1, L2 and L3, the points set on the apex of the equilateral triangle, L4 and L5, are stable centers shown in Fig.3. It gathers natural objects like asteroid called Trojan Asteroids which have been found located at Lagrange points 4 and 5 in various Sun-planet system. They are gathered by the gravitational force and remain in certain region because of stability at the Lagrange points. Jupiter-Trojan system is well-known for its large asteroid populations. These asteroids are often grouped into two camps: the Greek camp (L5) and the Trojan camp (L4), shown in Fig.6. The discoveries of these asteroids have a significant impact on the application of Lagrange points, suggesting that these points are validated in space observation rather than a pure mathematical implication.

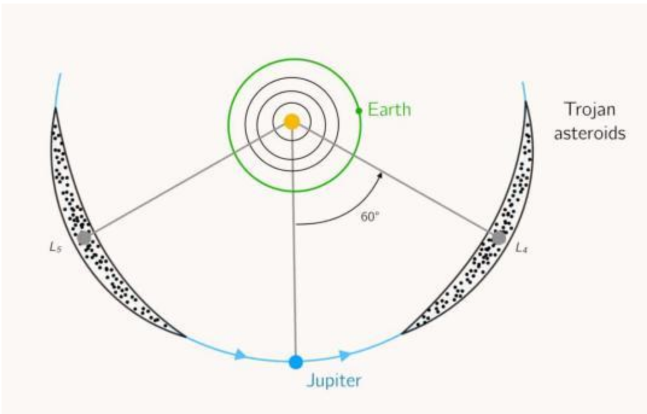

Figure 6. Earth, Jupiter, and satellites [11]

Despite the accumulation of these asteroids seems potentially dangerous to it host planet, dynamic structure of the Lagrange points allows for a process of orbital divergence, meaning that these asteroids will not drift toward to the host planet, serving as a safeguard mechanism.

Indeed, Lagrange points have satisfying condition for many practical physics experiments in space research. A nicknamed project LAGRANGE features in the detection and observation of gravitational wave, using laser interferometry. This project is conducted with three spacecrafts being placed on the Lagrange points L3, L4 and L5 of the Earth-moon (E-M) system since this the most stable geocentric configuration that has an average arm length of 670,000 km [12]. A different experiment with the same nickname LAGRANGE proposes using Lagrange points to exploit the time delay of electromagnetic signals traveling between Lagrangian points to measure relativistic effects, particularly those related to the Sun's gravitomagnetic field. It proposed that Lagrange points can allows measurements of the time delay in the propagation of electromagnetic wave caused by the gravitational field. Configuration at L1 and L2 would first allows a direct measurement (in the field of the Earth) of the overall delay on their propagation, as produced by the combined action of gravitomagnetic fields and oblateness, in addition to Shapiro time-delay [13]. The set up of the L4 and L5 Lagrangian, in an Earth-Moon, system would avoid the influence of the solar corona (the hot outer layer of the Sun) and the additional delay caused by the Sun's quadrupole moment.

4. Conclusion

The Circular Restricted Three-Body Problem (CR3BP) is a simplified model that is employed to describe the way celestial bodies orbit in systems like the Moon-Earth or Sun-Earth. The Lagrange points represent the positions where two dominant body forces and a weaker third body force balance each other out. The positions provide optimal chances for space science and research experiments. The L1, L2, L4, and L5 positions have been extremely useful in various space missions, especially in observing the Earth, Sun, and other space phenomena. The stable positions L4 and L5 are perfect for long-term monitoring, as evidenced by the Trojan asteroid cluster, and can be utilized to study gravitational waves and the gravitomagnetic field measurement. The unstable positions L1, L2, and L3, however, have aided in fundamental solar wind measurements, climate fluctuations, and space weather studies but need periodic adjustments because these spaces are unstable. Lagrange points assist us in understanding gravity, placing satellites in space, and learning about space weather. They are necessary for science and space exploration. With advances in satellite technology and computer simulations, Lagrange points will continue to be helpful for numerous research endeavors, such as observing Earth, venturing into deep space, and conducting physics experiments. What it discovers at Lagrange points will influence future space explorations and provide new opportunities for fundamental physics and applied space technology.

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